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Jean-Philippe Garnier, Kazuo Nishimura, Alain Venditti

GREQAM

Groupement de Recherche en Economie
Quantitative d'Aix-Marseille - UMR-CNRS 6579
Ecole des Hautes Etudes en Sciences Sociales
Universités d'Aix-Marseille II et III

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Jean-Philippe GARNIER
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Intertemporal substitution in consumption, labor supply elasticity and sunspot fluctuations in continuous-time models*

Jean-Philippe GARNIER

Université de la Méditerranée, GREQAM, Marseille, France

Kazuo NISHIMURA

Institute of Economic Research, Kyoto University, Japan

and

Alain VENDITTI

CNRS - GREQAM, Marseille, France

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Abstract: *The aim of this paper is to discuss the roles of the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply on the local determinacy properties of the steady state in a two-sector economy with CES technologies and sector-specific externalities. Our main results provide necessary and sufficient conditions for local indeterminacy. First we show that the consumption good sector needs to be capital intensive at the private level and labor intensive at the social level. Second, we prove that under this capital intensity configuration, the existence of sunspot fluctuations is obtained if and only if the elasticity of intertemporal substitution in consumption is large enough but the elasticity of the labor supply is low enough. In particular, we will show on the one hand that when the labor supply is infinitely elastic, the steady state is always saddle-point stable, and on the other hand that when the elasticity of intertemporal substitution in consumption is infinite, labor does not have any influence on the local stability properties of the equilibrium path.*

Keywords: *Sector-specific externalities, constant returns, capital-labor substitution, intertemporal substitution in consumption, elastic labor, indeterminacy.*

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1 Introduction

Since Benhabib and Farmer [1, 2], it is common to assume a large elasticity of intertemporal substitution in consumption and a large elasticity of the labor supply to obtain local indeterminacy of equilibria in infinite-horizon growth models with productive externalities and increasing returns at the social level.[1] Basically, an infinite elasticity of labor is often considered. It is important to point out that all these results are established either within one-sector growth models,[2] or within two-sector growth models with identical technologies at the private level.[3]

Considering instead a two-sector model with different Cobb-Douglas technologies at the private level characterized by sector-specific externalities and constant social returns, Benhabib and Nishimura [4] provide conclusions which suggest that the above common practice has to be more carefully examined. They assume a separable utility function which is linear with respect to consumption and strictly concave with respect to labor. The elasticity of intertemporal substitution in consumption is therefore infinite while the elasticity of labor is finite. They prove the following two fundamental results:

i) local indeterminacy arises even under constant returns provided there is a reversal of factor intensities between the private and the social levels, the consumption good being privately capital intensive but socially labor intensive;

ii) the occurrence of local indeterminacy is obtained without particular restriction concerning the elasticity of labor.

This last point is even reinforced by Benhabib, Nishimura and Venditti [5] who show that in a similar two-sector model but with inelastic labor, the technological mechanism based on the broken duality between Rybczynski and Stolper-Samuelson effects is sufficient to generate sunspot fluctuations. Such a conclusion is puzzling when compared to the basic results initially

[1] See also the survey of Benhabib and Farmer [3] for a large list of references.

[2] See for instance Lloyd-Braga, Nourry and Venditti [10], Pintus [12].

[3] See for instance Harrison [8], Harrison and Weder [9].

formulated by Benhabib and Farmer [1, 2] under increasing social returns.

Our aim in this paper is to understand this puzzle and thus to study carefully the role of labor on the existence of sunspot fluctuations in two-sector models with constant social returns. To do so, we believe that the best strategy is to start from the beginning by considering a general formulation without any a priori. We thus assume that the economy is composed by two sectors, one producing a pure consumption good, the other producing an investment good, characterized by CES technologies with asymmetric elasticities of capital-labor substitution which can take any positive value.[4] The externalities are sector-specific and the returns to scale are constant at the social level, thus decreasing at the private level.[5] On the preference side, we assume a CES additively separable utility function defined over consumption and labor. The elasticity of intertemporal substitution in consumption and the elasticity of the labor supply can take any value between 0 and $+\infty$.

Our main results provide necessary and sufficient conditions for local indeterminacy. First we show that the consumption good sector needs to be capital intensive at the private level and labor intensive at the social level. Second, we prove that under this capital intensity configuration, the existence of sunspot fluctuations is obtained if and only if, as it is commonly accepted, the elasticity of intertemporal substitution in consumption is large enough but, contrary to the standard belief, the elasticity of the labor supply is low enough. In particular, we show on the one hand that when the elasticity of intertemporal substitution in consumption is infinite, labor does not have any influence on the local stability properties of the equilibrium path, and on the other hand that when the labor supply is infinitely elastic, the steady state is always saddle-point stable. Actually, we prove that the consideration of endogenous labor does not introduce any additional room for sunspot fluctuations.

[4]The Cobb-Douglas formulation considered by Benhabib and Nishimura [4] or Benhabib, Nishimura and Venditti [5] will be treated as a particular case.

[5]We shall consider small externalities so that the private returns are only slightly decreasing. This assumption can be justified by the existence of a fixed factor, such as land, which is not accumulated and normalized to 1.

The paper is organized as follows: Section 2 presents the basic model, the intertemporal equilibrium, the steady state and the characteristic polynomial. The main results are exposed in Section 3. Some concluding comments are provided in Section 4 and a final Appendix contains all the proofs.

2 The model

2.1 The production structure

We consider an economy producing a pure consumption good y_0 and a pure capital good y_1. Each good is assumed to be produced by capital x_{1j} and labor x_{0j}, $j = 0, 1$, through a CES technology which contains sector specific externalities. The representative firm in each industry indeed faces the following technology, called *private production function*:

$$y_j = \left(\beta_{0j} x_{0j}^{-\rho_j} + \beta_{1j} x_{1j}^{-\rho_j} + e_j(X_{0j}, X_{1j})\right)^{-1/\rho_j}, \quad j = 0, 1 \tag{1}$$

with $\beta_{ij} > 0$, $\rho_j > -1$ and $\varsigma_j = 1/(1 + \rho_j) \geq 0$ the elasticity of capital-labor substitution. The positive externalities are equal to

$$e_j(X_{0j}, X_{1j}) = b_{0j} X_{0j}^{-\rho_j} + b_{1j} X_{1j}^{-\rho_j}$$

with $b_{ij} \geq 0$ and X_{ij} denoting the average use of input i in sector j. We assume that these economy-wide averages are taken as given by each individual firm. At the equilibrium, since all firms of sector j are identical, we have $X_{ij} = x_{ij}$ and we may define the *social production functions* as follows

$$y_j = \left(\hat{\beta}_{0j} x_{0j}^{-\rho_j} + \hat{\beta}_{1j} x_{1j}^{-\rho_j}\right)^{-1/\rho_j} \tag{2}$$

with $\hat{\beta}_{ij} = \beta_{ij} + b_{ij}$. The returns to scale are therefore constant at the social level, and decreasing at the private level. We assume that in each sector $j = 0, 1$, $\hat{\beta}_{0j} + \hat{\beta}_{1j} = 1$ so that the production functions collapse to Cobb-Douglas in the particular case $\rho_j = 0$. Total labor is given by $\ell = x_{00} + x_{01}$, and the total stock of capital is given by $x_1 = x_{10} + x_{11}$.

Choosing the consumption good as the numeraire, i.e. $p_0 = 1$, a firm in each industry maximizes its profit given the output price p_1, the rental rate of capital w_1 and the wage rate w_0. The first order conditions subject to the private technologies (1) give

$$x_{ij}/y_j = (p_j \beta_{ij}/w_i)^{\frac{1}{1+\rho_j}} \equiv a_{ij}(w_i, p_j), \quad i, j = 0, 1 \tag{3}$$

We call a_{ij} the input coefficients from the *private* viewpoint. If the agents take account of externalities as endogenous variables in profit maximization, the first order conditions subject to the social technologies (2) give on the contrary

$$x_{ij}/y_j = \left(p_j\hat{\beta}_{ij}/w_i\right)^{1/(1+\rho_j)} \equiv \bar{a}_{ij}(w_i, p_j), \quad i, j = 0, 1 \tag{4}$$

We call $\bar{a}_{ij}$ the input coefficients from the *social* viewpoint. We also define

$$\hat{a}_{ij}(w_i, p_j) \equiv (\hat{\beta}_{ij}/\beta_{ij})a_{ij}(w_i, p_j) \tag{5}$$

as the *quasi* input coefficients from the *social* viewpoint, and it is easy to derive that

$$\hat{a}_{ij}(w_i, p_j) = \bar{a}_{ij}(w_i, p_j)\left(\hat{\beta}_{ij}/\beta_{ij}\right)^{\rho_j/(1+\rho_j)}$$

Notice that $\hat{a}_{ij} = \bar{a}_{ij}$ if there is no externality coming from input i in sector j, i.e. $b_{ij} = 0$, or if the production function is Cobb-Douglas, i.e. $\rho_j = 0$. As we will show below, the factor-price frontier, which gives a relationship between input prices and output prices, is not expressed with the input coefficients from the social viewpoint but with the quasi input coefficients from the social viewpoint.

Based on these input coefficients it may be shown that, as in the case with symmetric elasticities of capital-labor substitution,[6] the factor-price frontier is determined by the quasi input coefficients from the social viewpoint while the factor market clearing equation depends on the input coefficients from the private perspective:[7]

Lemma 1. *Denote* $p = (1, p_1)'$, $w = (w_0, w_1)'$ *and* $\hat{A}(w, p) = [\hat{a}_{ij}(w_i, p_j)]$. *Then* $p = \hat{A}'(w, p)w$.

Lemma 2. *Denote* $x = (\ell, x_1)'$, $y = (y_0, y_1)'$ *and* $A(w, p) = [a_{ij}(w_i, p_j)]$. *Then* $A(w, p)y = x$.

Note that at the equilibrium, the wage rate and the rental rate are functions of the output price only, i.e. $w_0 = w_0(p_1)$, $w_1 = w_1(p_1)$, while outputs are functions of the capital stock, total labor and the output price, $y_j = \tilde{y}_j(x_1, \ell, p_1)$, $j = 0, 1$.

[6] See Nishimura and Venditti [11].

[7] See Garnier, Nishimura and Venditti [6] for the proofs of these results.

2.2 Intertemporal equilibrium and steady state

The economy is populated by a large number of identical infinitely-lived agents. We assume without loss of generality that the total population is constant and normalized to one. At each period a representative agent supplies elastically an amount of labor $\ell \in (0, \bar{\ell})$, with $\bar{\ell} > 0$ (possibly infinite) his endowment of labor. He then derives utility from consumption c and leisure $\mathcal{L} = \bar{\ell} - \ell$ according to the following function

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{\ell^{1+\gamma}}{1+\gamma}$$

with $\sigma \geq 0$ and $\gamma \geq 0$. The elasticity of intertemporal substitution in consumption is thus given by $\epsilon_c = 1/\sigma$ while the elasticity of the labor supply is given by $\epsilon_\ell = 1/\gamma$. Considering the external effects (e_0, e_1) as given, profit maximization in both sectors described in Section 2.1 gives demands for capital and labor as functions of the capital stock, the production level of the investment good, total labor and the external effects, namely $\tilde{x}_{ij} = x_{ij}(x_1, y_1, \ell, e_0, e_1)$, $i, j = 0, 1$. The production frontier is then defined as

$$c = T(x_1, y_1, \ell, e_0, e_1) = \left(\beta_{00}\tilde{x}_{00}^{-\rho_0} + \beta_{10}\tilde{x}_{10}^{-\rho_0} + e_0\right)^{-1/\rho_0}$$

From the envelope theorem we easily get $w_1 = T_1(x_1, y_1, \ell, e_0, e_1)$, $p_1 = -T_2(x_1, y_1, \ell, e_0, e_1)$ and $w_0 = T_3(x_1, y_1, \ell, e_0, e_1)$.

The intertemporal optimization problem of the representative agent can be described as:

$$\begin{aligned}
\max_{\{x_1(t), y_1(t), \ell(t)\}} \quad & \int_0^{+\infty} \left[\frac{T(x_1(t), y_1(t), \ell(t), e_0(t), e_1(t))^{1-\sigma}}{1-\sigma} - \frac{\ell(t)^{1+\gamma}}{1+\gamma}\right] e^{-\delta t} dt \\
s.t. \quad & \dot{x}_1(t) = y_1(t) - g x_1(t) \\
& x_1(0) \ given \\
& \{e_j(t)\}_{t \geq 0}, \ j = 0, 1, \ given
\end{aligned} \tag{6}$$

where $\delta \geq 0$ is the discount rate and $g > 0$ is the depreciation rate of the capital stock. We can write the modified Hamiltonian in current value as:

$$\mathcal{H} = \frac{T(x_1(t), y_1(t), \ell(t), e_0(t), e_1(t))^{1-\sigma}}{1-\sigma} - \frac{\ell(t)^{1+\gamma}}{1+\gamma} + q_1(t)\left(y_1(t) - g x_1(t)\right)$$

with $q_1(t) = e^{\delta t} p_1(t)$. The necessary conditions which describe the solution to problem (6) are therefore given by the following equations:

$$\begin{aligned} q_1(t) &= p_1(t)c(t)^{-\sigma} \qquad (7)\\ \ell(t)^\gamma &= w_0 c(t)^{-\sigma} \qquad (8)\\ \dot{x}_1(t) &= y_1(t) - g x_1(t) \qquad (9)\\ \dot{q}_1(t) &= (\delta+g)q_1(t) - w_1(t)c(t)^{-\sigma} \qquad (10)\end{aligned}$$

As shown in Section 2.1, we have $w_0 = w_0(p_1)$ and $c = \tilde{y}_0(x_1, \ell, p_1) = T(x_1, \tilde{y}_1(x_1, \ell, p_1), \ell, e_0(x_1, \ell, p_1), e_1(x_1, \ell, p_1))$. Therefore, solving equation (8) describing the labor-leisure trade-off at the equilibrium, we may express the labor supply as a function of the capital stock and the output price, $\ell = \ell(x_1, p_1)$. Then, we get $y_0 = c(x_1, p_1) \equiv \tilde{y}_0(x_1, \ell(x_1, p_1), p_1)$ and $y_1 = y_1(x_1, p_1) \equiv \tilde{y}_1(x_1, \ell(x_1, p_1), p_1)$. Considering (7)-(10) and denoting $E(x_1, p_1) \equiv 1 - \sigma(p_1/c)(\partial c/\partial p_1)$, the equations of motion are derived as

$$\begin{aligned} \dot{x}_1 &= y_1(x_1, p_1) - g x_1 \\ \dot{p}_1 &= \tfrac{1}{E(x_1,p_1)}\left[(\delta+g)p_1 - w_1(p_1) + \sigma \tfrac{p_1}{c}\left(y_1(x_1,p_1) - g x_1\right)\right] \end{aligned} \qquad (11)$$

Any solution $\{x_1(t), p_1(t)\}_{t\geq 0}$ that also satisfies the transversality condition

$$\lim_{t\to+\infty} e^{-\delta t} p_1(t) x_1(t) = 0$$

is called an equilibrium path.

A steady state is defined by a pair (x_1^*, p_1^*) solution of

$$\begin{aligned} y_1(x_1, p_1) &= g x_1 \\ w_1(p_1) &= (\delta+g)p_1 \end{aligned} \qquad (12)$$

We introduce the following restriction on parameters' values:

Assumption 1. *$\beta_{11} > \delta + g$ and $\rho_1 \in (\hat{\rho}_1, +\infty)$ with*

$$\hat{\rho}_1 \equiv \frac{ln\hat{\beta}_{11}}{ln\left(\frac{\beta_{11}}{\delta+g}\right) - ln\hat{\beta}_{11}} \in (-1, 0) \qquad (13)$$

Considering the fact that, within continuous-time models, the discount rate δ and the capital depreciation rate g are quite small, the restriction $\beta_{11} > \delta + g$ does not appear to be too demanding. Under Assumption 1 we derive positiveness and interiority of the steady state values for input demand functions x_{ij} and we get the following result:

Proposition 1. *Under Assumption 1, there exists a unique steady state $(x_1^*, p_1^*) > 0$.*

2.3 Characteristic polynomial

Linearizing the dynamical system (11) around (x_1^*, p_1^*) gives:

$$J = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} - g & \frac{\partial y_1}{\partial p_1} \\ \frac{\sigma}{E}\frac{p_1^*}{c^*}\frac{\partial c}{\partial x_1}\left(\frac{\partial y_1}{\partial x_1} - g\right) & \frac{1}{E}\left[\delta + g - \frac{\partial w_1}{\partial p_1} + \sigma\frac{p_1^*}{c^*}\frac{\partial c}{\partial x_1}\frac{\partial y_1}{\partial p_1}\right] \end{pmatrix} \tag{14}$$

As we show in Appendix 5.2, all these partial derivatives are functions of σ and γ. The role of γ of course occurs through the presence of endogenous labor but remains implicit at that stage mainly because of our methodology to derive the dynamical system (11) from the first order conditions (7)-(10).

Any solution from (11) that converges to the steady state (x_1^*, p_1^*) satisfies the transversality condition and is an equilibrium. Therefore, given $x_1(0)$, if there is more than one initial price $p_1(0)$ in the stable manifold of (x_1^*, p_1^*), the equilibrium path from $x_1(0)$ will not be unique. In particular, if J has two eigenvalues with negative real parts, there will be a continuum of converging paths and thus a continuum of equilibria.

Definition 1. *If the locally stable manifold of the steady state (x_1^*, p_1^*) is two-dimensional, then (x_1^*, p_1^*) is said to be locally indeterminate.*

The eigenvalues of J are given by the roots of the following characteristic polynomial

$$\mathcal{P}(\lambda) = \lambda^2 - \mathcal{T}\lambda + \mathcal{D} \tag{15}$$

with

$$\begin{aligned} \mathcal{D}(\sigma,\gamma) &= \frac{1}{E}\left(\frac{\partial y_1}{\partial x_1} - g\right)\left(\delta + g - \frac{\partial w_1}{\partial p_1}\right) \\ \mathcal{T}(\sigma,\gamma) &= \frac{1}{E}\left\{\frac{\partial y_1}{\partial x_1} + \delta - \frac{\partial w_1}{\partial p_1} + \sigma\frac{p_1^*}{c^*}\left[\frac{\partial c}{\partial x_1}\frac{\partial y_1}{\partial p_1} - \frac{\partial c}{\partial p_1}\left(\frac{\partial y_1}{\partial x_1} - g\right)\right]\right\} \end{aligned} \tag{16}$$

(See Appendix 5.2 for the detailed expressions of these derivatives). Local indeterminacy requires therefore that $\mathcal{D}(\sigma,\gamma) > 0$ and $\mathcal{T}(\sigma,\gamma) < 0$.

3 Main results

Our main objective is to study jointly the roles of the elasticity of intertemporal substitution in consumption and the elasticity of the labor supply on the local determinacy properties of the long-run equilibrium. We will first consider three polar cases from which we will be able to derive conclusions for the general configuration with $\sigma, \gamma > 0$. In the first case we assume a linear utility with respect to consumption, i.e. an infinite elasticity of intertemporal substitution in consumption. Then we consider the second extreme configuration which is based on a linear utility function with recpect to labor, i.e. an infinite elasticity of the labor supply. Finally, the third particular case concerns the model with inelastic labor and non-linear utility function with respect to consumption.

3.1 Local indeterminacy with an infinite elasticity of intertemporal substitution in consumption: $\sigma = 0$, $\gamma > 0$

In the case of a linear utility function with respect to consumption, i.e. $\sigma = 0$, we get $E = 1$, $\mathcal{D}(0, \gamma) = \lambda_1 \lambda_2$ and $\mathcal{T}(0, \gamma) = \lambda_1 + \lambda_2$ with

$$\lambda_1 = \left(\frac{\partial y_1}{\partial x_1} - g \right), \quad \lambda_2 = \left(\delta + g - \frac{\partial w_1}{\partial p_1} \right) \tag{17}$$

the two eigenvalues and

$$\frac{dy_1}{dx_1} = \frac{a_{00}}{a_{11}a_{00} - a_{10}a_{01}}, \quad \frac{dw_1}{dp_1} = \frac{\hat{a}_{00}}{\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01}} \tag{18}$$

as shown in Appendix 5.2.

Using the definitions of input coefficients given in Section 2.1 allows to characterize the elements of $\partial y_1 / \partial x_1$ and $\partial w_1 / \partial p_1$ in terms of the capital intensity differences across sectors at the private and quasi-social levels:

Definition 2. *The consumption good is said to be:*

i) capital intensive at the ***private*** *level if and only if* $a_{11}a_{00} - a_{10}a_{01} < 0$,

ii) ***quasi*** *capital intensive at the* ***social*** *level if and only if* $\hat{a}_{11}\hat{a}_{00} - \hat{a}_{10}\hat{a}_{01} < 0$.

We may conveniently relate these input coefficients to the CES parameters:[8]

Proposition 2. *Let Assumption 1 hold. At the steady state:*

i) the consumption good is capital (labor) intensive from the private perspective if and only if

[8]See Garnier, Nishimura and Venditti [6] for a proof of Proposition 2.

$$b \equiv 1 - \left(\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right)^{\frac{1}{1+\rho_0}} \left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}} - \hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{\rho_1-\rho_0}{\rho_1(1+\rho_0)}} < (>) 0$$

ii) the consumption good is quasi capital (labor) intensive from the social perspective if and only if

$$\hat{b} \equiv 1 - \frac{\hat{\beta}_{10}\hat{\beta}_{01}}{\hat{\beta}_{00}\hat{\beta}_{11}} \left(\frac{\beta_{00}\beta_{11}}{\beta_{10}\beta_{01}}\right)^{\frac{\rho_0}{1+\rho_0}} \left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}} - \hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{\rho_1-\rho_0}{\rho_1(1+\rho_0)}} < (>) 0$$

It follows that that $\partial y_1/\partial x_1$ corresponds to the factor intensity difference from the private viewpoint, and is associated with Rybczynski effects, while $\partial w_1/\partial p_1$ corresponds to the quasi factor intensity difference from the social viewpoint, and is associated with Stolper-Samuelson effects.[9]

A first general conclusion is exhibited:

Proposition 3. *Under Assumption 1, let* $\sigma = 0$. *Then the local stability properties of the steady state only depend on the CES coefficients* $(\beta_{ij}, b_{ij}, \rho_i)$, $i = 0, 1$, *and do not depend on the elasticity of the labor supply* $\epsilon_\ell = 1/\gamma$.

This result explains why in Benhabib and Nishimura [4] the consideration of endogenous labor in a two-sector model with Cobb-Douglas technologies, sector specific externalities and linear utility in consumption leads to some conditions for local indeterminacy which are only based on capital intensity differences at the private and social levels and which do not depend on the elasticity of the labor supply.

Considering Definition 2, Proposition 2 with (17) and (18), we derive that local indeterminacy requires a factor intensities reversal between the private and the quasi social levels.[10]

Proposition 4. *Under Assumption 1, the steady state is locally indeterminate if and only if the consumption good is capital intensive from the private perspective* $(b < 0)$, *but quasi labor intensive from the social perspective* $(\hat{b} > 0)$.

[9]See Appendix 5.2.

[10]A similar conclusion has been provided by Nishimura and Venditti [11] in a two-sector continuous-time model with CES technologies characterized by symmetric elasticities of capital-labor substitution.

Notice from Proposition 2 that the capital intensity differences at the private and quasi social levels are linked as follows:

$$\hat{b} = 1 - \frac{\frac{\hat{\beta}_{10}\hat{\beta}_{01}}{\hat{\beta}_{11}\hat{\beta}_{00}}}{\frac{\beta_{10}\beta_{01}}{\beta_{11}\beta_{00}}}(1-b) \tag{19}$$

It follows that when $b < 0$, a necessary condition for $\hat{b} > 0$ is given by the following Assumption:

Assumption 2. $\frac{\hat{\beta}_{10}\hat{\beta}_{01}}{\hat{\beta}_{11}\hat{\beta}_{00}} < \frac{\beta_{10}\beta_{01}}{\beta_{11}\beta_{00}}$

Under Assumption 2, Garnier, Nishimura and Venditti [6] provide conditions on the sectoral elasticities of capital-labor substitution to get locally indeterminate equilibria.

3.2 Local determinacy with an infinite elasticity of labor supply : $\sigma > 0$, $\gamma = 0$

Consider a linear utility function with respect to labor, i.e. $\gamma = 0$. As shown in Lloyd-Braga, Nourry and Venditti [10] or Pintus [12], the occurrence of local indeterminacy in aggregate models requires the consideration of large (close to infinite) elasticities of labor supply. The following Theorem shows on the contrary that within two-sector models, the conclusion is completely reversed.

Theorem 1. *Under Assumption 1, if $\gamma = 0$ the steady state is saddle-point stable.*

The proof is based on the fact that for any given $\sigma > 0$

$$\lim_{\gamma \to 0} \mathcal{D}(\sigma, \gamma) = \mathcal{D}(\sigma, 0) < 0$$

This result implies that for any intertemporal elasticity of substitution in consumption and any amount of externalities, the steady state is locally determinate.

Theorem 1 needs also to be compared with the main results of Benhabib and Farmer [2]. Indeed, they consider a two-sector model with endogenous labor and Cobb-Douglas technologies characterized by increasing social returns derived from sector-specific externalities. They assume that both technologies are identical at the private level, i.e. $b = 0$ or equivalently $a_{11}a_{00} - a_{10}a_{01} = 0$. They show that local indeterminacy occurs under a large elasticity of the labor supply while our Theorem 1 provides a completely opposite conclusion. Such a drastic difference between one-sector models or two-sector models with identical private technologies on one side, and two-sector models with different private technologies on the other side seems to be explained both by the different assumptions on returns to scale at the social level, and, as it is shown in Appendix 5.2, by the fact that if $b = 0$, i.e. $a_{11}a_{00} - a_{10}a_{01} = 0$, then the partial derivatives entering the Jacobian matrix (14) are no longer well-defined.

3.3 Local indeterminacy with inelastic labor supply: $\sigma > 0$, $\gamma = +\infty$

When the utility function is non-linear with respect to consumption, i.e. $\sigma > 0$, and labor is inelastic, i.e. $\gamma = +\infty$, the utility function becomes

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

A simple geometrical methodology allows to provide a clear picture of the local determinacy properties of equilibria. As in Grandmont, Pintus and De Vilder [7], we analyze the local stability of the steady state by studying the variations of $\mathcal{T}(\sigma, +\infty)$ and $\mathcal{D}(\sigma, +\infty)$ in the $(\mathcal{T}, \mathcal{D})$ plane when the inverse of the elasticity of intertemporal substitution in consumption σ varies continuously. Solving the two equations in (16) with respect to σ shows that when σ covers the interval $[0, +\infty)$, $\mathcal{D}(\sigma, +\infty)$ and $\mathcal{T}(\sigma, +\infty)$ vary along a line, called in what follows Δ_∞, which is defined by the following equation

$$\mathcal{D} = \mathcal{S}_\infty \mathcal{T} + \frac{\left(\frac{\partial y_1}{\partial x_1} - g\right)\left(\delta + g - \frac{\partial w_1}{\partial p_1}\right)\left[\frac{\partial c}{\partial x_1}\frac{\partial y_1}{\partial p_1} - \frac{\partial c}{\partial p_1}\left(\frac{\partial y_1}{\partial x_1} - g\right)\right]}{\frac{\partial c}{\partial x_1}\frac{\partial y_1}{\partial p_1} + \frac{\partial c}{\partial p_1}\left(\delta + g - \frac{\partial w_1}{\partial p_1}\right)} \tag{20}$$

with

$$\mathcal{S}_\infty = \frac{\frac{\partial c}{\partial p_1}\left(\frac{\partial y_1}{\partial x_1} - g\right)\left(\delta + g - \frac{\partial w_1}{\partial p_1}\right)}{\frac{\partial c}{\partial x_1}\frac{\partial y_1}{\partial p_1} + \frac{\partial c}{\partial p_1}\left(\delta + g - \frac{\partial w_1}{\partial p_1}\right)} \tag{21}$$

the slope of Δ_∞. Notice that that since $\gamma = +\infty$, labor is inelastic and $\ell^* = 1$. Moreover, (x_1^*, p_1^*) does not depend on σ, as shown in Proposition 1. The steady state therefore remains the same along the line Δ_∞.

As σ increases from 0 to $+\infty$ the pair $(\mathcal{T}(\sigma,+\infty),\mathcal{D}(\sigma,+\infty))$ moves along Δ_∞ which is characterized by the starting point $(\mathcal{T}(0,+\infty),\mathcal{D}(0,+\infty))$ and the end point $(\mathcal{T}(+\infty,+\infty),\mathcal{D}(+\infty,+\infty))$, such that:

$$\begin{array}{l} \mathcal{T}(0,+\infty) = \frac{\partial y_1}{\partial x_1} + \delta - \frac{\partial w_1}{\partial p_1},\ \mathcal{D}(0,+\infty) = \left(\frac{\partial y_1}{\partial x_1} - g\right)\left(\delta + g - \frac{\partial w_1}{\partial p_1}\right) \\ \mathcal{T}(+\infty,+\infty) = -\frac{\frac{\partial c}{\partial x_1}\frac{\partial y_1}{\partial p_1}}{\frac{\partial c}{\partial p_1}} + \left(\frac{\partial y_1}{\partial x_1} - g\right),\ \mathcal{D}(+\infty,+\infty) = 0 \end{array} \tag{22}$$

(See Appendix 5.2 for detailed expressions). Therefore Δ_∞ is a segment which starts in $(\mathcal{T}(0,+\infty),\mathcal{D}(0,+\infty))$ and ends in $(\mathcal{T}(+\infty,+\infty),0)$, i.e. on the abscissa axis. Notice that $(\mathcal{T}(0,+\infty),\mathcal{D}(0,+\infty))$ corresponds to the characteristic roots obtained in Section 3.2 under $\sigma = 0$. Proposition 4 gives necessary and sufficient conditions for the occurrence of local indeterminacy when $\sigma = 0$, namely $b < 0$ and $\hat{b} > 0$. We start by proving that when $\sigma > 0$ and $\gamma = +\infty$, these restrictions are still necessary conditions for the existence of locally indeterminate equilibria.

Proposition 5. *Under Assumption 1, let $\sigma > 0$ and $\gamma = +\infty$. Then the following results hold:*

i) When $b < 0$ and $\hat{b} < 0$, or $b > 0$ and $\hat{b} > 0$, the steady state is saddle-point stable for any $\sigma > 0$.

ii) When $b > 0$ and $\hat{b} < 0$, there exists $\bar{\sigma} \in (0,+\infty)$ such that the steady state is locally unstable when $\sigma \in [0,\bar{\sigma})$ and saddle-point stable when $\sigma > \bar{\sigma}$.

As a result, we introduce the following restrictions:

Assumption 3. *The consumption good is capital intensive from the private perspective ($b < 0$), but quasi labor intensive from the social perspective ($\hat{b} > 0$).*

Under this Assumption the starting point is such that $\mathcal{D}(0,+\infty) > 0$ and $\mathcal{T}(0,+\infty) < 0$. Based on these results we finally need to study how $\mathcal{D}(\sigma,+\infty)$ and $\mathcal{T}(\sigma,+\infty)$ vary with σ.

Lemma 3. *Under Assumptions 1-3, $\mathcal{D}(\sigma,+\infty)$ and $\mathcal{T}(\sigma,+\infty)$ are respectively increasing and decreasing functions of σ, and there exists $\bar{\sigma} \in (0,+\infty)$ such that $\mathcal{D}(\sigma,+\infty) > 0$ when $\sigma \in [0,\bar{\sigma})$, $\lim_{\sigma\to\bar{\sigma}^-}\mathcal{D}(\sigma,+\infty) = +\infty$,*

$\lim_{\sigma\to\bar{\sigma}^+} \mathcal{D}(\sigma,+\infty) = -\infty$, *and* $\mathcal{D}(\sigma,+\infty) < 0$ *when* $\sigma > \bar{\sigma}$. *Moreover,* $\mathcal{S}_\infty < 0$.

The critical bound $\bar{\sigma}$ in Proposition 5 and Lemma 3 is defined from (16) as the value of σ such that $E = 0$, and is equal to

$$\bar{\sigma} = \frac{c^*}{p_1^*(\partial c/\partial p_1)} \tag{23}$$

Starting from $(\mathcal{T}(0,+\infty),\mathcal{D}(0,+\infty))$ with $\mathcal{D}(0,+\infty) > 0$ and $\mathcal{T}(0,+\infty) < 0$, the point $(\mathcal{T}(\sigma,+\infty),\mathcal{D}(\sigma,+\infty))$ increases along a line Δ_∞ as $\sigma \in (0,\bar{\sigma})$, goes through $\mathcal{D}(\sigma,+\infty) = +\infty$ as $\sigma = \bar{\sigma}$ and finally increases from $\mathcal{D}(\sigma,+\infty) = -\infty$ as $\sigma > \bar{\sigma}$ until it reaches the end point $(\mathcal{T}(+\infty,+\infty),0)$, as shown in the following Figure:

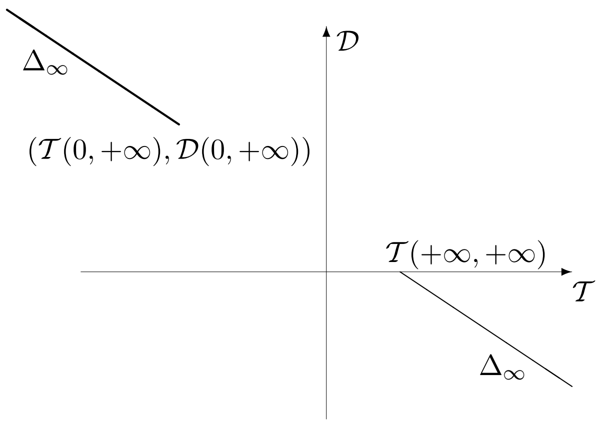

Figure 1: Local indeterminacy with non linear utility.

Note that the sign of $\mathcal{T}(+\infty,+\infty)$ does not have any influence on the next Theorem which follows from Proposition 5, Lemma 3 and Figure 1:

Theorem 2. *Under Assumption 1, let* $\gamma = +\infty$. *Then there exists* $\bar{\sigma} \in (0,+\infty)$ *such that the steady state is locally indeterminate if and only if Assumptions 2-3 hold and* $\sigma \in [0,\bar{\sigma})$.

Benhabib and Nishimura [4] show numerically that in a two-sector model with Cobb-Douglas technologies, local indeterminacy can only occur with sufficiently low values of σ. We provide here a formal proof of a general result that holds for any values of the elasticities of capital-labor substitution satisfying Assumption 3.

3.4 Local indeterminacy with elastic labor and intertemporal substitution in consumption: $\sigma > 0$, $\gamma > 0$

When the utility function is non-linear with respect to consumption and labor, i.e. $\sigma > 0$ and $\gamma > 0$, we may again follow the simple geometrical methodology as in Grandmont, Pintus and De Vilder [7] to analyze the local stability of the steady state. For a given value of $\sigma > 0$, we now study the variations of $\mathcal{T}(\sigma,\gamma)$ and $\mathcal{D}(\sigma,\gamma)$ in the $(\mathcal{T},\mathcal{D})$ plane when the inverse of the elasticity of intertemporal substitution in consumption γ varies continuously. Solving the two equations in (16) with respect to γ shows that when γ covers the interval $[0,+\infty)$, $\mathcal{D}(\sigma,\gamma)$ and $\mathcal{T}(\sigma,\gamma)$ vary along a line, called in what follows $\Delta_\gamma(\sigma)$, which is defined by the following equation

$$\mathcal{D} = \mathcal{S}_\gamma(\sigma)\mathcal{T} + \mathcal{M}_\gamma(\sigma) \tag{24}$$

with $\mathcal{S}_\gamma(\sigma)$ the slope and $\mathcal{M}_\gamma(\sigma)$ the constant term, both defined in Appendix 5.6. As in the previous Section, since $\gamma \in (0,+\infty)$, we have to compute the starting and end points of the pair $(\mathcal{T}(\sigma,\gamma),\mathcal{D}(\sigma,\gamma))$. We know that the starting point, corresponding to the case of inelastic labor with $\gamma = +\infty$, is defined as $(\mathcal{T}(\sigma,+\infty),\mathcal{D}(\sigma,+\infty))$ and necessarily belongs to the line Δ_∞ analyzed in Section 3.3. Similarly, the end point, corresponding to the case of infinitely elastic labor with $\gamma = 0$, is defined as $(\mathcal{T}(\sigma,0),\mathcal{D}(\sigma,0))$ and necessarily satisfies $\mathcal{D}(\sigma,0) < 0$ as shown in Section 3.2.

As in Section 3.3, we first prove that when $\sigma > 0$ and $\gamma > 0$, Assumption 3 is still a necessary condition for the existence of local indeterminacy.

Proposition 6. *Under Assumption 1, let $\sigma > 0$ and $\gamma > 0$ and consider the critical value $\bar{\sigma}$ as defined by (23). Then the following results hold:*

i) When $b < 0$ and $\hat{b} < 0$, or $b > 0$ and $\hat{b} > 0$, the steady state is saddle-point stable for any $\sigma > 0$ and $\gamma > 0$.

ii) When $b > 0$, $\hat{b} < 0$ and $\sigma > \bar{\sigma}$, the steady state is saddle-point stable for any $\gamma > 0$.

iii) When $b > 0$, $\hat{b} < 0$ and $\sigma \in [0,\bar{\sigma})$, there exists $\underline{\gamma}(\sigma) \in (0,+\infty)$ such that the steady state is locally unstable when $\gamma > \underline{\gamma}(\sigma)$ and saddle-point stable when $\gamma \in [0,\underline{\gamma}(\sigma))$.

In order to locate the line $\Delta_\gamma(\sigma)$ under Assumption 3, we finally need to study how $\mathcal{D}et(\sigma,\gamma)$ and $\mathcal{T}r(\sigma,\gamma)$ vary with γ, and the slope $\mathcal{S}_\gamma(\sigma)$.

Lemma 4. *Under Assumption 1-3, consider the critical value $\bar{\sigma}$ as defined by (23). Then there exist $\sigma_{\mathcal{D}} \in (\bar{\sigma}, +\infty)$ and $\sigma_{\mathcal{T}} \in (\bar{\sigma}, +\infty]$ such that the following results hold:*

i) $\mathcal{D}_2(\sigma,\gamma) < 0$ when $\sigma \in [0, \sigma_{\mathcal{D}})$, $\mathcal{D}_2(\sigma,\gamma) = 0$ when $\sigma = \sigma_{\mathcal{D}}$ and $\mathcal{D}_2(\sigma,\gamma) > 0$ when $\sigma > \sigma_{\mathcal{D}}$;

ii) $\mathcal{T}_2(\sigma,\gamma) > 0$ when $\sigma \in [0, \sigma_{\mathcal{T}})$, $\mathcal{T}_2(\sigma,\gamma) = 0$ when $\sigma = \sigma_{\mathcal{T}}$ and $\mathcal{T}_2(\sigma,\gamma) < 0$ when $\sigma > \sigma_{\mathcal{T}}$;

iii) $\mathcal{S}_\sigma(\gamma) < 0$ when $\sigma \in [0, \sigma^*)$ with $\sigma^* = \min\{\sigma_{\mathcal{D}}, \sigma_{\mathcal{T}}\} > \bar{\sigma}$.

In order to complete the geometrical analysis, we derive the following result:

Lemma 5. *Under Assumption 1-3, consider the critical value $\bar{\sigma}$ as defined by (23). Then:*

i) for any given $\sigma \geq \bar{\sigma}$, $\mathcal{D}(\sigma,\gamma) < 0$ for all $\gamma \geq 0$;

ii) for any given $\sigma \in [0, \bar{\sigma})$ there exists $\underline{\gamma}(\sigma) \in (0, +\infty)$ such that $\mathcal{D}(\sigma,\gamma) > 0$ when $\gamma > \underline{\gamma}(\sigma)$, $\lim_{\gamma \to \underline{\gamma}(\sigma)^+} \mathcal{D}(\sigma,\gamma) = +\infty$, $\lim_{\gamma \to \underline{\gamma}(\sigma)^-} \mathcal{D}(\sigma,\gamma) = -\infty$ and $\mathcal{D}(\sigma,\gamma) < 0$ when $\gamma \in [0, \underline{\gamma}(\sigma))$.

The critical bound $\underline{\gamma}(\sigma)$ in Proposition 6 and Lemma 5 is obviously a function of σ and is defined as the value of γ such that the denominator of $\mathcal{D}(\sigma,\gamma)$ is equal to zero. It is given by expression (54) in Appendix 5.9.

We immediately derive from Lemma 5 that the consideration of endogenous labor does not provide any additional room for the occurrence of local indeterminacy. Indeed, the existence of multiple equilibria again requires σ to be lower than the bound $\bar{\sigma}$ exhibited in the case of inelastic labor. Let us then consider a given $\sigma \in [0, \bar{\sigma})$. Lemma 4 shows that $\mathcal{D}(\sigma,\gamma)$ is a decreasing function of γ while $\mathcal{T}(\sigma,\gamma)$ is an increasing function of γ. Moreover, the slope $\mathcal{S}_\sigma(\gamma)$ is negative. When $\gamma = +\infty$, the starting point is located on the line Δ_∞ associated with the case of inelastic labor. As γ is decreased, the pair $(\mathcal{T}(\sigma,\gamma), \mathcal{D}(\sigma,\gamma))$ moves upward along a line $\Delta_\gamma(\sigma)$, with $\mathcal{D}(\sigma,\gamma)$ going through $+\infty$ as $\gamma = \underline{\gamma}(\sigma)$, and finally coming from $\mathcal{D}(\sigma,\gamma) = -\infty$ as $\gamma > \underline{\gamma}(\sigma)$ until it reaches the end point $(\mathcal{T}(\sigma,0), \mathcal{D}(\sigma,0))$ characterized by $\mathcal{D}(\sigma,0) < 0$. All these results may be summarized by the following Figure:

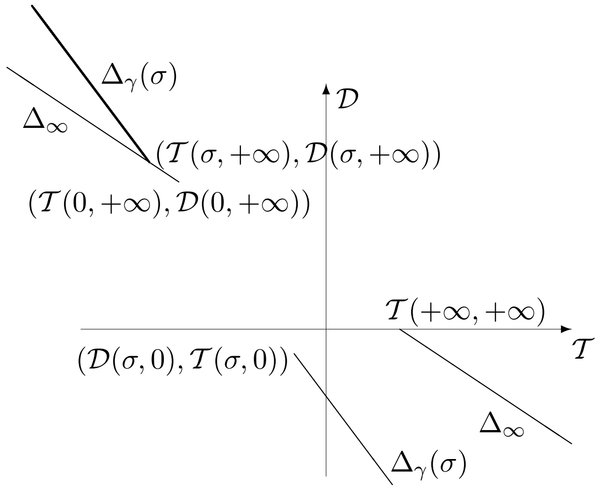

Figure 2: Local indeterminacy with endogenous labor.

We then derive from Proposition 6, Lemmas 4, 5 and Figure 4:

Theorem 3. *Under Assumption 1, consider the critical value $\bar{\sigma}$ as defined by (23) and the bound $\underline{\gamma}(\sigma)$ derived in Lemma 5. Then, the steady state is locally indeterminate if and only if Assumptions 2-3 hold, $\sigma \in [0, \bar{\sigma})$ and $\gamma > \underline{\gamma}(\sigma)$.*

As shown by expression (55) in Appendix 5.9, for any given $\sigma \in [0, \bar{\sigma})$, $\underline{\gamma}(\sigma)$ is an increasing function of σ. We then derive the following Corollary:

Corollary 1. *Under Assumption 1-3, consider the critical value $\bar{\sigma}$ as defined by (23) and let $\sigma \in [0, \bar{\sigma})$. Then, when a lower elasticity of intertemporal substitution in consumption is considered, local indeterminacy requires a lower elasticity of the labor supply, i.e. for any $\sigma_1, \sigma_2 \in [0, \bar{\sigma})$ such that $\sigma_1 > \sigma_2$, we have $\underline{\gamma}(\sigma_1) > \underline{\gamma}(\sigma_2)$*

Notice finally that a direct consequence of Propositions 5-6 and Theorems 2-3 is that a Hopf bifurcation cannot occur in this model.[11]

[11] Within a continuous time framework, a Hopf bifurcation occurs if $\mathcal{D} > 0$ and $\mathcal{T} = 0$. Propositions 5-6 and Theorems 2-3 show that when $\sigma \in [0, \bar{\sigma})$, the segments Δ_∞ and $\Delta_\gamma(\sigma)$ cannot intersect the ordinate axis corresponding to $\mathcal{D} > 0$ and $\mathcal{T} = 0$.

4 Concluding comments

We have considered a two-sector economy with CES technologies containing sector-specific externalities and additively separable CES preferences defined over consumption and leisure. We have provided necessary and sufficient conditions for local indeterminacy. First we have shown that as in the case with inelastic labor and a linear utility function, the consumption good sector needs to be capital intensive at the private level and labor intensive at the social level. Second, we have proved that under this capital intensity configuration, the existence of sunspot fluctuations is obtained if and only if the elasticity of intertemporal substitution in consumption is large enough but the elasticity of the labor supply is low enough. In particular, we have shown that when the labor supply is infinitely elastic, the steady state is always saddle-point stable.

5 Appendix

5.1 Proof of Proposition 1

Maximizing the profit subject to the private technologies (1) gives the first order conditions

$$p_j \beta_{ij} \left(y_j / x_{ij} \right)^{1+\rho_j} = w_i, \quad i, j = 0, 1 \tag{25}$$

Considering the steady state with $y_1 = gx_1$ and $w_1 = (\delta + g)p_1$, we get

$$x_{11} = \left(\frac{\beta_{11}}{\delta + g} \right)^{\frac{1}{1+\rho_1}} gx_1 \tag{26}$$

Using the social production function (2) for the investment good we derive

$$x_{01} = \left(\frac{\left(\frac{\beta_{11}}{\delta+g} \right)^{\frac{\rho_1}{1+\rho_1}} - \hat{\beta}_{11}}{\hat{\beta}_{01}} \right)^{-\frac{1}{\rho_1}} \left(\frac{\beta_{11}}{\delta + g} \right)^{\frac{1}{1+\rho_1}} gx_1$$

and thus

$$\frac{x_{01}}{x_{11}} = \left(\frac{\left(\frac{\beta_{11}}{\delta+g} \right)^{\frac{\rho_1}{1+\rho_1}} - \hat{\beta}_{11}}{\hat{\beta}_{01}} \right)^{-\frac{1}{\rho_1}} \tag{27}$$

Finally we easily obtain from (25):

$$\begin{aligned} \frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}} &= \left(\frac{x_{01}}{x_{11}}\right)^{1+\rho_1}\left(\frac{x_{10}}{x_{00}}\right)^{1+\rho_0} \\ \Leftrightarrow \quad \frac{x_{10}}{x_{00}} &= \left(\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right)^{\frac{1}{1+\rho_0}}\left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}}-\hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{1+\rho_1}{\rho_1(1+\rho_0)}} \end{aligned} \tag{28}$$

Considering (27), (28) and $x_{00}+x_{01}=\ell$, $x_1 = x_{10}+x_{11}$, we get

$$x_1^* \quad = \quad \frac{\ell^*\left(\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right)^{\frac{1}{1+\rho_0}}\left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}}-\hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{1+\rho_1}{\rho_1(1+\rho_0)}}}{1-\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{1}{1+\rho_1}}gb} \equiv \ell^*\kappa^*$$

with b given in Proposition 2. Equation (25) for $i=1$ and $j=0$ gives

$$w_1 = \beta_{10}\left(\hat{\beta}_{00}\left(\frac{x_{10}}{x_{00}}\right)^{\rho_0}+\hat{\beta}_{10}\right)^{-\frac{1+\rho_0}{\rho_0}} \tag{29}$$

Considering (28) and the fact that $w_1 = (\delta+g)p_1$ implies

$$p_1^* \quad = \quad \frac{\beta_{10}}{\delta+g}\left[\hat{\beta}_{00}\left(\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right)^{\frac{\rho_0}{1+\rho_0}}\left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}}-\hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{\rho_0(1+\rho_1)}{\rho_1(1+\rho_0)}}+\hat{\beta}_{10}\right]^{-\frac{1+\rho_0}{\rho_0}}$$

The substitution of (26) and (28) into (2) gives the expression of c^*, namely

$$c^* = \ell^*\kappa^*\left[1-\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{1}{1+\rho_1}}g\right]\left[\hat{\beta}_{10}+\hat{\beta}_{00}\left(\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right)^{\frac{\rho_0}{1+\rho_0}}\left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}}-\hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{\rho_0(1+\rho_1)}{\rho_1(1+\rho_0)}}\right]^{-\frac{1}{\rho_0}}$$

$$\equiv \ell^*\chi^*$$

Finally, using (25) we derive

$$w_0 = w_1\frac{\beta_{01}}{\beta_{11}}\left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}}-\hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{1+\rho_1}{\rho_1}} \tag{30}$$

and substituting c^* and (30) into (8) gives the stationary labor supply

$$\ell^* = \left\{\frac{\frac{\beta_{10}\beta_{01}}{\beta_{11}}\left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}}-\hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{1+\rho_1}{\rho_1}}}{\left[1-\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{1}{1+\rho_1}}g\right]^{\sigma}\left[\hat{\beta}_{10}+\hat{\beta}_{00}\left(\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right)^{\frac{\rho_0}{1+\rho_0}}\left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}}-\hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{\rho_0(1+\rho_1)}{\rho_1(1+\rho_0)}}\right]^{\frac{1+\rho_0-\sigma}{\rho_0}}\kappa^{*\sigma}}\right\}^{\frac{1}{\gamma+\sigma}}$$

□

5.2 Computation of $\mathcal{D}(\sigma,\gamma)$ and $\mathcal{T}(\sigma,\gamma)$

Consider the expressions (16). We need therefore to compute the following four derivatives: $\partial c/\partial x_1$, $\partial c/\partial p_1$, $\partial y_1/\partial p_1$ and $\partial w_1/\partial p_1$. We proceed from total differenciation of the factor-price frontier and the factor market clearing equation given in Lemmas 1-2. From the factor-price frontier and the fact that the function $\hat{A}(w,p)$ is homogeneous of degree zero in w and p, we immediately derive:

$$\tfrac{dw_1}{dp_1} = \tfrac{\hat{a}_{00}}{\hat{a}_{11}\hat{a}_{00}-\hat{a}_{10}\hat{a}_{01}} \quad \text{and thus} \quad \tfrac{dw_0}{dp_1} = -\tfrac{\hat{a}_{10}}{\hat{a}_{11}\hat{a}_{00}-\hat{a}_{10}\hat{a}_{01}} \tag{31}$$

Notice that these derivatives correspond to the Stolper-Samuelson effects.

Consider now Lemma 2 with (3):

$$\begin{array}{rcl} a_{00}(w_0,1)y_0 + a_{01}(w_0,p)y_1 & = & \ell \\ a_{10}(w_1,1)y_0 + a_{11}(w_1,p)y_1 & = & x_1 \end{array} \tag{32}$$

Total differenciation gives:

$$\begin{array}{rcl} a_{00}dy_0 + a_{01}dy_1 + \frac{\partial a_{00}}{\partial w_0}y_0dw_0 + y\left(\frac{\partial a_{01}}{\partial w_0}dw_0 + \frac{\partial a_{01}}{\partial p_1}dp_1\right) & = & d\ell \\ a_{10}dy_0 + a_{11}dy_1 + \frac{\partial a_{10}}{\partial w_1}y_0dw_1 + y\left(\frac{\partial a_{11}}{\partial w_1}dw_1 + \frac{\partial a_{11}}{\partial p_1}dp_1\right) & = & dx_1 \end{array} \tag{33}$$

Let us start with dc/dx_1 and dy_1/dx_1 by considering $y_0 = c$ and $dw_0 = dw_1 = dp_1 = 0$. We get

$$\begin{array}{rcl} \frac{dc}{dx_1} & = & -\frac{a_{01}}{a_{11}a_{00}-a_{10}a_{01}} + \frac{a_{11}}{a_{11}a_{00}-a_{10}a_{01}}\frac{d\ell}{dx_1} \\ \frac{dy_1}{dx_1} & = & \frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}} - \frac{a_{10}}{a_{11}a_{00}-a_{10}a_{01}}\frac{d\ell}{dx_1} \end{array} \tag{34}$$

Notice that when labor is inelastic, i.e. $\gamma = +\infty$, or when the utility function is linear with respect to consumption, i.e. $\sigma = 0$, then $d\ell/dx_1 = 0$ (see (40) below), and these derivatives correspond to the Rybczynski effects.

To compute dc/dp_1 and dy_1/dp_1, we set $dx_1 = 0$ and we have to consider the factor-price frontier given by Lemma 1. Solving for w_0 and w_1 gives:

$$w_0 = \tfrac{\hat{a}_{10}}{\hat{a}_{10}\hat{a}_{01}-\hat{a}_{11}\hat{a}_{00}}\left(p_1 - \tfrac{\hat{a}_{11}}{\hat{a}_{10}}\right), \quad w_1 = \tfrac{\hat{a}_{00}}{\hat{a}_{11}\hat{a}_{00}-\hat{a}_{10}\hat{a}_{01}}\left(p_1 - \tfrac{\hat{a}_{01}}{\hat{a}_{00}}\right) \tag{35}$$

We also have to consider from (31) that

$$dw_0 = \tfrac{\hat{a}_{10}}{\hat{a}_{10}\hat{a}_{01}-\hat{a}_{11}\hat{a}_{00}}dp_1, \quad dw_1 = \tfrac{\hat{a}_{00}}{\hat{a}_{11}\hat{a}_{00}-\hat{a}_{10}\hat{a}_{01}}dp_1$$

From (3) we finally derive

$$\tfrac{\partial a_{i0}(w_i,1)}{\partial w_i} = -\tfrac{a_{i0}}{w_i(1+\rho_0)}, \quad \tfrac{\partial a_{i1}(w_i,p_1)}{\partial w_i} = -\tfrac{a_{i1}}{w_i(1+\rho_1)}, \quad \tfrac{\partial a_{i1}(w_i,p_1)}{\partial p_1} = \tfrac{a_{i1}}{p_1(1+\rho_1)}$$

Substituting all this into (33) with $dx_1 = 0$, while considering (32) gives:

$$\begin{aligned} a_{00}dc + a_{01}dy_1 + dp_1 \left[\frac{1+\rho_1 a_{00}c+\rho_0 a_{01}y_1}{(1+\rho_0)(1+\rho_1)} \frac{\hat{a}_{10}}{\hat{a}_{11}-\hat{a}_{10}p_1} + \frac{a_{01}}{1+\rho_1}\frac{y_1}{p_1}\right] &= d\ell \\ a_{10}dc + a_{11}dy_1 + dp_1 \left[\frac{x_1+\rho_1 a_{10}c+\rho_0 a_{11}y_1}{(1+\rho_0)(1+\rho_1)} \frac{\hat{a}_{00}}{\hat{a}_{01}-\hat{a}_{00}p_1} + \frac{a_{11}}{1+\rho_1}\frac{y_1}{p_1}\right] &= 0 \end{aligned} \tag{36}$$

From (5), we get:

$$\frac{\hat{a}_{00}}{\hat{a}_{00}p_1-\hat{a}_{01}} = \frac{1}{p_1}\left(1 - \frac{\hat{\beta}_{01}}{\hat{\beta}_{00}}\frac{a_{00}^{\rho_0}}{a_{01}^{\rho_1}}\right)^{-1}, \quad \frac{\hat{a}_{10}}{\hat{a}_{10}p_1-\hat{a}_{11}} = \frac{1}{p_1}\left(1 - \frac{\hat{\beta}_{11}}{\hat{\beta}_{10}}\frac{a_{10}^{\rho_0}}{a_{11}^{\rho_1}}\right)^{-1} \tag{37}$$

Solving system (36) taking into account Proposition 1 and the fact that at the steady state $y_1 = gx_1$ then leads to:

$$\begin{aligned} \frac{dy_1}{dp_1} &= \frac{a_{00}\ell^*}{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)p_1^*}\Bigg\{ \left(1 - \frac{\hat{\beta}_{01}}{\hat{\beta}_{00}}\frac{a_{00}^{\rho_0}}{a_{01}^{\rho_1}}\right)^{-1}(\kappa^* + \rho_1 a_{10}\chi^* + \rho_0 a_{11}g\kappa^*) \\ &+ \left(\frac{\hat{\beta}_{11}}{\hat{\beta}_{10}}\frac{a_{10}^{\rho_0}}{a_{11}^{\rho_1}} - 1\right)^{-1}\frac{a_{10}}{a_{00}}(1+\rho_1 a_{00}\chi^* + \rho_0 a_{01}g\kappa^*)\Bigg\} - \frac{g\ell^*\kappa^*}{(1+\rho_1)p_1^*} \\ &- \frac{a_{10}}{a_{11}a_{00}-a_{10}a_{01}}\frac{d\ell}{dp_1} \\ \frac{dc}{dp_1} &= \frac{-a_{01}\ell^*}{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)p_1^*}\Bigg\{ \left(1 - \frac{\hat{\beta}_{01}}{\hat{\beta}_{00}}\frac{a_{00}^{\rho_0}}{a_{01}^{\rho_1}}\right)^{-1}(\kappa^* + \rho_1 a_{10}\chi^* + \rho_0 a_{11}g\kappa^*) \\ &+ \left(\frac{\hat{\beta}_{11}}{\hat{\beta}_{10}}\frac{a_{10}^{\rho_0}}{a_{11}^{\rho_1}} - 1\right)^{-1}\frac{a_{11}}{a_{01}}(1+\rho_1 a_{00}\chi^* + \rho_0 a_{01}g\kappa^*)\Bigg\} + \frac{a_{11}}{a_{11}a_{00}-a_{10}a_{01}}\frac{d\ell}{dp_1} \end{aligned} \tag{38}$$

From (28) and (29) we get

$$\begin{aligned} w_1 &= \beta_{10}\left(\hat{\beta}_{00}\left(\frac{\beta_{10}\beta_{01}}{\beta_{00}\beta_{11}}\right)^{\frac{\rho_0}{1+\rho_0}}\left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}} - \hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{\rho_0(1+\rho_1)}{\rho_1(1+\rho_0)}} + \hat{\beta}_{10}\right)^{-\frac{1+\rho_0}{\rho_0}} \\ &\equiv \beta_{10}\mathcal{C}^{-\frac{1+\rho_0}{\rho_0}} \end{aligned}$$

Substituting this expression and (30) into (3) and considering Proposition 2 gives:

$$\begin{aligned} 1 - \frac{\hat{\beta}_{11}}{\hat{\beta}_{10}}\frac{a_{10}^{\rho_0}}{a_{11}^{\rho_1}} &= 1 - \frac{\hat{\beta}_{11}}{\hat{\beta}_{10}}\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{-\rho_1}{1+\rho_1}}\mathcal{C} \\ &\equiv 1 - \mathcal{A} = -\frac{\hat{b}}{1-\hat{b}}\left[1 - \hat{\beta}_{11}\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{-\rho_1}{1+\rho_1}}\right] \\ 1 - \frac{\hat{\beta}_{01}}{\hat{\beta}_{00}}\frac{a_{00}^{\rho_0}}{a_{01}^{\rho_1}} &= 1 - \frac{\hat{\beta}_{01}}{\hat{\beta}_{00}}\left(\frac{\beta_{00}\beta_{11}}{\beta_{10}\beta_{01}}\right)^{\frac{\rho_0}{1+\rho_0}}\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{-\rho_1}{1+\rho_1}}\left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}} - \hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{\rho_1-\rho_0}{\rho_1(1+\rho_0)}}\mathcal{C} \\ &\equiv 1 - \mathcal{B} = 1 - \mathcal{A}(1-\hat{b}) = \hat{b}\hat{\beta}_{11}\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{-\rho_1}{1+\rho_1}} \end{aligned} \tag{39}$$

We have finally to derive the expressions of $d\ell/dx_1$ and $d\ell/dp_1$. Total differenciation of equation (8) gives:

$$\gamma d\ell = \frac{\ell}{w_0}\frac{dw_0}{dp_1}dp_1 - \sigma\frac{\ell}{c}\left(\frac{dc}{dx_1}dx_1 + \frac{dc}{dp_1}dp_1\right)$$

Assume first that $dp_1 = 0$. From (34) with Proposition 1, we derive:

$$\frac{d\ell}{dx_1} = \frac{\frac{a_{01}\sigma}{(a_{11}a_{00}-a_{10}a_{01})\gamma\chi^*}}{1+\frac{a_{11}\sigma}{(a_{11}a_{00}-a_{10}a_{01})\gamma\chi^*}} \tag{40}$$

Assume finally that $dx_1 = 0$. Equations (31), (35) and (37) give

$$\frac{1}{w_0}\frac{dw_0}{dp_1} = \frac{1}{p_1(1-\mathcal{A})}$$

Therefore, considering equation (38) with Proposition 1, we derive:

$$\frac{d\ell}{dp_1} = \frac{\frac{\sigma}{\gamma}\frac{\ell^*}{p_1^*}\left[\frac{1}{\sigma(1-\mathcal{A})}+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)\chi^*}\right]}{1+\frac{a_{11}\sigma}{(a_{11}a_{00}-a_{10}a_{01})\gamma\chi^*}} \tag{41}$$

with

$$\mathcal{Z}_1 = \frac{\kappa^*+\rho_1 a_{10}\chi^*+\rho_0 a_{11} g\kappa^*}{1-\mathcal{B}}, \quad \mathcal{Z}_2 = \frac{a_{11}}{a_{01}}\frac{1+\rho_1 a_{00}\chi^*+\rho_0 a_{01} g\kappa^*}{\mathcal{A}-1} \tag{42}$$

Substituting (34), (38), (40) and (41) into (16) finally gives after straightforward but tedious computations:

$$\begin{aligned}
\mathcal{D}(\sigma,\gamma) &= \frac{\left[\frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}}-g+\frac{(1-ga_{11})\sigma}{(a_{11}a_{00}-a_{10}a_{01})\gamma\chi^*}\right]\left(\delta+g-\frac{dw_1}{dp_1}\right)}{1+\frac{\sigma}{a_{11}a_{00}-a_{10}a_{01}}\left[\frac{a_{11}}{\gamma\chi^*}\left(1-\frac{1}{1-\mathcal{A}}\right)+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{\chi^*(1+\rho_0)(1+\rho_1)}\right]} \\
\mathcal{T}(\sigma,\gamma) &= \left\{\frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}} + \delta - \frac{dw_1}{dp_1} + \frac{\frac{\sigma}{\gamma\chi^*}\left[(1-ga_{11})\left(1-\frac{1}{1-\mathcal{A}}\right)+a_{11}\left(\delta+g-\frac{dw_1}{dp_1}\right)\right]}{a_{11}a_{00}-a_{10}a_{01}}\right. \\
&\quad \left. + \frac{\sigma}{\chi^*}\frac{a_{01}}{a_{11}}\frac{[\mathcal{Z}_2+(1+\rho_0)a_{11}g\kappa^*-ga_{11}(\mathcal{Z}_1+\mathcal{Z}_2)]}{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)}\right\} \\
&\quad \times \left[1+\frac{\sigma}{a_{11}a_{00}-a_{10}a_{01}}\left[\frac{a_{11}}{\gamma\chi^*}\left(1-\frac{1}{1-\mathcal{A}}\right)+\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{\chi^*(1+\rho_0)(1+\rho_1)}\right]\right]^{-1}
\end{aligned} \tag{43}$$

□

5.3 Proof of Theorem 1

From (43) we derive for any given $\sigma > 0$:

$$\lim_{\gamma\to 0}\mathcal{D}(\sigma,\gamma) \equiv \mathcal{D}(\sigma,0) = \frac{(1-ga_{11})\left(\delta+g-\frac{dw_1}{dp_1}\right)}{a_{11}\left(1-\frac{1}{1-\mathcal{A}}\right)}$$

From Lemma 2, $x_1 = a_{10}y_0 + a_{11}y_1$. Since $y_1 = gx_1$ at the steady state we get

$$a_{10}y_0 + ga_{11}x_1 = x_1 \quad \Leftrightarrow \quad a_{10}y_0 = (1-ga_{11})x_1 > 0 \tag{44}$$

Notice now that

$$a_{01} = \left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{1}{1+\rho_1}} \left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}} - \hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{\frac{-1}{\rho_1}}, \quad a_{11} = \left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{1}{1+\rho_1}}$$

$$a_{00} = \left(\frac{\beta_{00}\beta_{11}}{\beta_{10}\beta_{01}}\right)^{\frac{1}{1+\rho_0}} \left(\frac{\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}} - \hat{\beta}_{11}}{\hat{\beta}_{01}}\right)^{-\frac{1+\rho_1}{\rho_1(1+\rho_0)}} \mathcal{C}^{\frac{1}{\rho_0}}, \quad a_{10} = \mathcal{C}^{\frac{1}{\rho_0}} \tag{45}$$

From (5) and Proposition 2 we get

$$\delta + g - \frac{dw_1}{dp_1} = -\frac{\delta+g}{\hat{\beta}_{11}\hat{b}}\left[\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}} - \hat{b}\hat{\beta}_{11}\right] \tag{46}$$

with $1 - \hat{b} > 0$. From (39) we finally get

$$1 - \frac{1}{1-\mathcal{A}} = \frac{\left[\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}} - \hat{b}\hat{\beta}_{11}\right]}{\hat{b}\left[\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}} - \hat{\beta}_{11}\right]} \tag{47}$$

Therefore we obtain from Assumption 1:

$$\mathcal{D}(\sigma, 0) = -\frac{1-ga_{11}}{a_{11}}\frac{\delta+g}{\hat{\beta}_{11}}\left[\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}} - \hat{\beta}_{11}\right] < 0 \tag{48}$$

□

5.4 Proof of Proposition 5

Consider the expression of $\mathcal{D}(\sigma, \gamma)$ in (43). We start by the following Lemma:

Lemma 6. *Under Assumption 1, if $b < 0$ and $\hat{b} < 0$, or $b > 0$ and $\hat{b} > 0$, then $\mathcal{D}(\sigma, \gamma) < 0$ for any $\sigma > 0$ and $\gamma > 0$.*

Proof: Assume first that $b < 0$ and $\hat{b} < 0$. We derive from Definition 2, Proposition 2 and equations (39), (42), (46), (47) that

$$a_{11}a_{00} - a_{10}a_{01} < 0, \quad \frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}} - g < 0, \quad \delta + g - \frac{dw_1}{dp_1} > 0, \quad 1 - \mathcal{A} > 0,$$

$$1 - \mathcal{B} < 0, \qquad 1 - \frac{1}{1-\mathcal{A}} < 0, \qquad \mathcal{Z}_1 < 0, \qquad \mathcal{Z}_2 < 0$$

Then $\mathcal{D}(\sigma, \gamma) < 0$ for any $\sigma, \gamma > 0$. Assume now that $b > 0$ and $\hat{b} > 0$. We derive from Definition 2, Proposition 2 and (39), (42), (46), (47) that

$$a_{11}a_{00} - a_{10}a_{01} > 0, \quad \frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}} - g > 0, \quad \delta + g - \frac{dw_1}{dp_1} < 0, \quad 1 - \mathcal{A} < 0,$$

$$1 - \mathcal{B} > 0, \qquad 1 - \frac{1}{1-\mathcal{A}} > 0, \qquad \mathcal{Z}_1 > 0, \qquad \mathcal{Z}_2 > 0$$

It follows again that $\mathcal{D}(\sigma, \gamma) < 0$ for any $\sigma > 0$ and $\gamma > 0$. □

We may now prove Proposition 5. If $\gamma = +\infty$, Lemma 6 implies that when $b < 0$ and $\hat{b} < 0$, or $b > 0$ and $\hat{b} > 0$, $\mathcal{D}(\sigma, +\infty) < 0$ for any $\sigma > 0$. Consider now the case with $b > 0$ and $\hat{b} < 0$. As shown in (40) and (41), when $\gamma = +\infty$, we have $d\ell/dx_1 = d\ell/dp_1 = 0$. From the expressions of $\mathcal{D}(\sigma, +\infty)$ and $\mathcal{T}(\sigma, +\infty)$ given in (16), we derive:

$$\begin{aligned} \mathcal{D}_1(\sigma, +\infty) &= \frac{\mathcal{D}(0,+\infty)}{E^2} \frac{\partial c}{\partial p_1} \frac{p_1^*}{c^*} \\ \mathcal{T}_1(\sigma, +\infty) &= \frac{p_1^*}{c^* E^2} \left[\frac{\partial c}{\partial x_1} \frac{\partial y_1}{\partial p_1} + \left(\delta + g - \frac{\partial w_1}{\partial p_1} \right) \frac{\partial c}{\partial p_1} \right] \end{aligned} \tag{49}$$

From (34), (38), (40), (42) and (45) we get

$$\begin{aligned} \frac{dy_1}{dp_1} &= \frac{dy_1}{dx_1} \frac{1}{(1+\rho_0)(1+\rho_1)} \frac{\ell^*}{p_1^*} \left(\mathcal{Z}_1 + \frac{a_{10}a_{01}}{a_{11}a_{00}} \mathcal{Z}_2 \right) - \frac{g\ell^*\kappa^*}{(1+\rho_1)p_1^*} \\ \frac{dc}{dp_1} &= \frac{dc}{dx_1} \frac{1}{(1+\rho_0)(1+\rho_1)} \frac{\ell^*}{p_1^*} (\mathcal{Z}_1 + \mathcal{Z}_2) \end{aligned} \tag{50}$$

Assuming $b > 0$ and $\hat{b} < 0$ implies that $\delta + g - \partial w_1/\partial p_1 > 0$, $dy_1/dx_1 > 0$, $dc/dx_1 < 0$, $Z_1 < 0$ and $Z_2 < 0$, and thus $dy_1/dp_1 < 0$, $dc/dp_1 > 0$, $\mathcal{D}_1(\sigma, +\infty) > 0$, $\mathcal{T}_1(\sigma, +\infty) > 0$ with $\mathcal{D}(0, +\infty) > 0$ and $\mathcal{T}(0, +\infty) > 0$. Finally, using (43) with $\gamma = +\infty$, we derive that there exists $\bar{\sigma} \in (0, +\infty)$ such that when $\sigma = \bar{\sigma}$, we have $E = 0$ and $\lim_{\sigma \to \bar{\sigma}^-} \mathcal{D}(\sigma, +\infty) = +\infty$ while $\lim_{\sigma \to \bar{\sigma}^+} \mathcal{D}(\sigma, +\infty) = -\infty$. The Δ_∞ line is thus a segment such that starting from $(\mathcal{T}(0, +\infty), \mathcal{D}(0, +\infty))$, the point $(\mathcal{T}(\sigma, +\infty), \mathcal{D}(\sigma, +\infty))$ increases along Δ_∞ as $\sigma \in (0, \bar{\sigma})$, goes through $\mathcal{D}(\sigma, +\infty) = +\infty$ as $\sigma = \bar{\sigma}$ and finally increases from $\mathcal{D}(\sigma, +\infty) = -\infty$ as $\sigma > \bar{\sigma}$ until it reaches $(\mathcal{T}(+\infty, +\infty), 0)$ located on the abscissa axis, as shown in the following Figure:

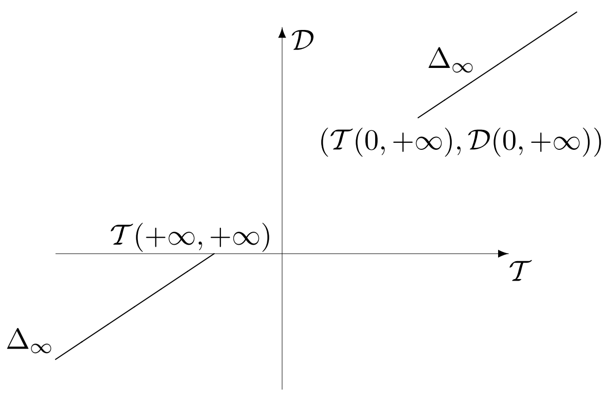

Figure 3: Saddle-point stability for large values of σ.

The result then follows. □

5.5 Proof of Lemma 3

Consider (49). Assumption 3 implies that $\mathcal{D}(0,+\infty) > 0$. It then follows from (39) that if $\hat{b} = 0$, $1-\mathcal{B} = 1-\mathcal{A} = 0$. Moreover, to get indeterminacy we have to assume $\hat{b} > 0$ so that $1-\mathcal{B} > 0$ and $1-\mathcal{A} < 0$. Consider now (50). Assumption 3 implies that $dy_1/dx_1 < 0$, $dc/dx_1 > 0$, $Z_1 > 0$ and $Z_2 > 0$, and thus $dy_1/dp_1 < 0$, $dc/dp_1 > 0$, $\mathcal{D}_1(\sigma,+\infty) > 0$ and $\mathcal{T}_1(\sigma,+\infty) < 0$. Moreover, there exists a critical value $\bar{\sigma}$ given by (23) such that when $\sigma = \bar{\sigma}$, we have $E = 0$. Substituting (34), (40) and (50) in (23) gives

$$\bar{\sigma} = \frac{\chi^*(a_{10}a_{01}-a_{11}a_{00})(1+\rho_0)(1+\rho_1)}{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)} \tag{51}$$

It follows that $\lim_{\sigma\to\bar{\sigma}^-} \mathcal{D}(\sigma,+\infty) = +\infty$ and $\lim_{\sigma\to\bar{\sigma}^+} \mathcal{D}(\sigma,+\infty) = -\infty$. Finally, since $\mathcal{S}_\infty = \mathcal{D}_1(\sigma,+\infty)/\mathcal{T}_1(\sigma,+\infty)$, we have $\mathcal{S}_\infty < 0$. □

5.6 Computation of $\Delta_\gamma(\sigma)$

Solving (43) with respect to $\sigma/[\gamma\chi^*(a_{11}a_{00} - a_{10}a_{01})]$ gives

$$\frac{\sigma/\gamma\chi^*}{a_{11}a_{00}-a_{10}a_{01}} = \frac{\left(\frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}}-g\right)\left(\delta+g-\frac{dw_1}{dp_1}\right)-\mathcal{D}\left[1+\frac{\sigma}{\chi^*}\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)}\right]}{\mathcal{D}a_{11}\left(1-\frac{1}{1-\mathcal{A}}\right)-(1-ga_{11})\left(\delta+g-\frac{dw_1}{dp_1}\right)}$$

$$\begin{aligned}\frac{\sigma/\gamma\chi^*}{a_{11}a_{00}-a_{10}a_{01}} &= \left\{\frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}} + \delta - \frac{dw_1}{dp_1} - \mathcal{T}\left[1+\frac{\sigma}{\chi^*}\frac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)}\right]\right. \\ &\quad \left. + \frac{\sigma}{\chi^*}\frac{a_{01}}{a_{11}}\frac{[\mathcal{Z}_2+(1+\rho_0)a_{11}g\kappa^*-ga_{11}(\mathcal{Z}_1+\mathcal{Z}_2)]}{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)}\right\} \\ &\quad \times \left[[\mathcal{T}a_{11}-(1-ga_{11})]\left(1-\frac{1}{1-\mathcal{A}}\right) - a_{11}\left(\delta+g-\frac{dw_1}{dp_1}\right)\right]^{-1}\end{aligned}$$

Equalizing these two expressions allows to define, for any given $\sigma \in (0,+\infty)$, a linear relationship between $\mathcal{D}$ and $\mathcal{T}$ as γ varies in $(0,+\infty)$, namely:

$$\mathcal{D} = \mathcal{S}_\gamma(\sigma)\mathcal{T} + \mathcal{M}_\gamma(\sigma)$$

with

$$\mathcal{S}_\gamma(\sigma) = \frac{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)\left(\delta+g-\frac{dw_1}{dp_1}\right)(\mathcal{Z}_1+\mathcal{Z}_2)(\sigma-\sigma_{\mathcal{D}})}{(\sigma/\chi^*)a_{01}(\sigma-\sigma_{\mathcal{T}})\left[a_{11}\left(\delta+g-\frac{dw_1}{dp_1}\right)(\mathcal{Z}_1+\mathcal{Z}_2)+\left(1-\frac{1}{1-\mathcal{A}}\right)[\mathcal{Z}_1-(1+\rho_0)a_{11}g\kappa^*]\right]}$$

$$\mathcal{M}_\gamma(\sigma) = \frac{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)\left(\delta+g-\frac{dw_1}{dp_1}\right)\left[\left(\frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}}-g\right)\frac{1-ga_{11}}{\mathcal{A}-1}+\left(\delta+g-\frac{dw_1}{dp_1}\right)\frac{a_{01}a_{10}}{a_{11}a_{00}-a_{10}a_{01}}\right]}{(\sigma/\chi^*)a_{01}(\sigma-\sigma_{\mathcal{T}})\left[a_{11}\left(\delta+g-\frac{dw_1}{dp_1}\right)(\mathcal{Z}_1+\mathcal{Z}_2)+\left(1-\frac{1}{1-\mathcal{A}}\right)[\mathcal{Z}_1-(1+\rho_0)a_{11}g\kappa^*]\right]}$$

$$\sigma_{\mathcal{D}} = \frac{\chi^*(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)\left[a_{11}\left(1-\frac{1}{1-\mathcal{A}}\right)\left(\frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}}-g\right)-(1-ga_{11})\right]}{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}$$

$$\sigma_{\mathcal{T}} = \frac{\chi^*(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)\left[\left(1-\frac{1}{1-\mathcal{A}}\right)\frac{a_{01}a_{10}}{a_{11}a_{00}-a_{10}a_{01}}-a_{11}\left(\delta+g-\frac{dw_1}{dp_1}\right)\right]}{a_{01}\left[a_{11}\left(\delta+g-\frac{dw_1}{dp_1}\right)(\mathcal{Z}_1+\mathcal{Z}_2)+\left(1-\frac{1}{1-\mathcal{A}}\right)[\mathcal{Z}_1-(1+\rho_0)a_{11}g\kappa^*]\right]}$$

□

5.7 Proof of Proposition 6

Result i) is an immediate consequence of Lemma 6. Consider then the case with $b > 0$ and $\hat{b} < 0$. We start by the following Lemma:

Lemma 7. *Under Assumption 1, consider the critical value $\bar{\sigma}$ as defined by (51). If $b > 0$ and $\hat{b} < 0$, the following results hold:*

i) when $\sigma > \bar{\sigma}$, then $\mathcal{D}(\sigma,\gamma) < 0$ for any $\gamma > 0$,

ii) when $\sigma \in (0,\bar{\sigma})$, then there exists $\underline{\gamma}(\sigma) > 0$ such that $\mathcal{D}(\sigma,\gamma) > 0$ for any $\gamma > \underline{\gamma}(\sigma)$, $\mathcal{D}(\sigma,\underline{\gamma}(\sigma)) = \pm\infty$ and $\mathcal{D}(\sigma,\gamma) < 0$ for any $\gamma \in (0,\underline{\gamma}(\sigma))$.

Proof: Assume that $b > 0$ and $\hat{b} < 0$ and consider the expression of $\mathcal{D}(\sigma,\gamma)$ in (43). We derive from Definition 2, Proposition 2 and equations (39), (42), (46), (47) that

$$\begin{array}{lll} a_{11}a_{00}-a_{10}a_{01} > 0, & \frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}} - g > 0, & \delta + g - \frac{dw_1}{dp_1} > 0, \\ 1-\mathcal{A} > 0, & 1-\mathcal{B} < 0, & 1 - \frac{1}{1-\mathcal{A}} < 0, \\ \mathcal{Z}_1 < 0, & \mathcal{Z}_2 < 0 & \end{array} \tag{52}$$

The result follows from the fact that the numerator of $\mathcal{D}(\sigma,\gamma)$ is positive while its denominator is positive when $\sigma \in (0,\bar{\sigma})$ and negative when $\sigma > \bar{\sigma}$. □

We may now prove Proposition 6. Let us denote by $\mathcal{N}$ the denominator of $\mathcal{D}(\sigma,\gamma)$ in (43). Tedious but straightforward computations give:

$$\begin{aligned} \mathcal{D}_2(\sigma,\gamma) &= \frac{-\sigma\left(\delta+g-\frac{dw_1}{dp_1}\right)\left[(1-ga_{11})\left[1+\frac{\sigma a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{\chi^*(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)}\right]\right.}{(a_{11}a_{00}-a_{10}a_{01})\gamma^2\chi^*\mathcal{N}^2} \\ &\quad - \frac{\left.a_{11}\left(1-\frac{1}{1-\mathcal{A}}\right)\left(\frac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}}-g\right)\right]}{(a_{11}a_{00}-a_{10}a_{01})\gamma^2\chi^*\mathcal{N}^2} \end{aligned} \tag{53}$$

Assume that $b > 0$ and $\hat{b} < 0$. We derive from Lemma 7 and (52) that when $\sigma \in (0,\bar{\sigma})$, then $\mathcal{D}_2(\sigma,\gamma) < 0$.

Finally, we know from (48) that $\mathcal{D}(\sigma,0)<0$, and from (43), (45), (46) and (47) we derive

$$\begin{aligned}\mathcal{T}(\sigma,0) &= \frac{(1-ga_{11})\left(1-\frac{1}{1-\mathcal{A}}\right)+a_{11}\left(\delta+g-\frac{dw_1}{dp_1}\right)}{\left(1-\frac{1}{1-\mathcal{A}}\right)} \\ &= \frac{\left[\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}}-\hat{b}\hat{\beta}_{11}\right]}{\hat{\beta}_{11}\hat{b}\left[\left(\frac{\beta_{11}}{\delta+g}\right)^{\frac{\rho_1}{1+\rho_1}}-\hat{\beta}_{11}\right]}\frac{\hat{\beta}_{11}(1+a_{11}\delta)-\beta_{11}}{\left(1-\frac{1}{1-\mathcal{A}}\right)}<0\end{aligned}$$

Let us then consider a given $\sigma\in[0,\bar{\sigma})$. When $\gamma=+\infty$, the starting point is located on the line Δ_∞ as given in Figure 4, which is associated with the case of inelastic labor. As γ is decreased, the pair $(\mathcal{T}(\sigma,\gamma),\mathcal{D}(\sigma,\gamma))$ moves upward along a line $\Delta_\gamma(\sigma)$, since $\mathcal{D}_2(\sigma,\gamma)<0$, with $\mathcal{D}(\sigma,\gamma)$ going through $+\infty$ as $\gamma=\underline{\gamma}(\sigma)$, and finally coming from $\mathcal{D}(\sigma,\gamma)=-\infty$ as $\gamma>\underline{\gamma}(\sigma)$ until it reaches the end point $(\mathcal{T}(\sigma,0),\mathcal{D}(\sigma,0))$ characterized by $\mathcal{D}(\sigma,0)<0$ and $\mathcal{T}(\sigma,0)<0$. Since $\Delta_\gamma(\sigma)$ is a straight line, it cannot cross the negative orthant with $\mathcal{T}(\sigma,0)<0$ and $\mathcal{D}(\sigma,0)<0$. All these results may be summarized by the following Figure:

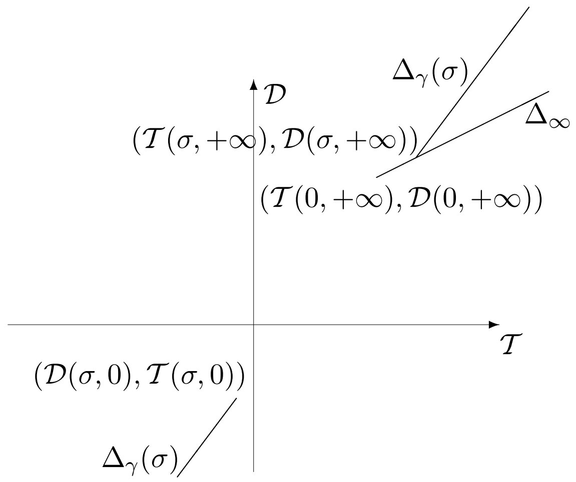

Figure 4: Saddle-point stability for low values of γ.

The result then follows. □

5.8 Proof of Lemma 4

Let us again denote by $\mathcal{N}$ the denominator of $\mathcal{D}(\sigma,\gamma)$ and $\mathcal{T}(\sigma,\gamma)$ in (43). Tedious but straightforward computations give:

$$\begin{aligned}\mathcal{T}_2(\sigma,\gamma) &= \tfrac{\sigma}{(a_{11}a_{00}-a_{10}a_{01})\gamma^2\chi^*\mathcal{N}^2}\left\{\tfrac{a_{01}a_{10}\left(1-\frac{1}{1-\mathcal{A}}\right)}{a_{11}a_{00}-a_{10}a_{01}} - \tfrac{a_{11}\left(\delta+g-\frac{dw_1}{dp_1}\right)}{1-\mathcal{A}}\right.\\ &- \tfrac{\sigma a_{01}}{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)\chi^*}\left[a_{11}(\mathcal{Z}_1+\mathcal{Z}_2)\left(\delta+g-\tfrac{dw_1}{dp_1}\right)\right.\\ &+ \left.\left.\left(1-\tfrac{1}{1-\mathcal{A}}\right)[\mathcal{Z}_1-(1+\rho_0)a_{11}g\kappa^*]\right]\right\}\end{aligned}$$

i) Under Assumption 3, we immediately derive from (53) that $\mathcal{D}_2(\sigma,\gamma) < 0$ if and only if $\sigma \in [0,\sigma_{\mathcal{D}})$ with $\sigma_{\mathcal{D}}$ given in Section 5.6 above. It is then easy to get that $\mathcal{D}_2(\bar{\sigma},\gamma) < 0$, with $\bar{\sigma}$ as defined by (51), and thus $\sigma_{\mathcal{D}} > \bar{\sigma}$.

ii) Let us denote

$$\begin{aligned}\mathcal{T}_2(\sigma,\gamma) &= \tfrac{\sigma}{(a_{11}a_{00}-a_{10}a_{01})\gamma^2\chi^*\mathcal{N}^2}\left\{\tfrac{a_{01}a_{10}\left(1-\frac{1}{1-\mathcal{A}}\right)}{a_{11}a_{00}-a_{10}a_{01}} - \tfrac{a_{11}\left(\delta+g-\frac{dw_1}{dp_1}\right)}{1-\mathcal{A}}\right.\\ &- \left.\tfrac{\sigma a_{01}}{(a_{11}a_{00}-a_{10}a_{01})(1+\rho_0)(1+\rho_1)\chi^*}\mathcal{Q}\right\}\end{aligned}$$

It follows that under Assumption 3, if $\mathcal{Q} \leq 0$ then $\mathcal{T}_2(\sigma,\gamma) > 0$ for all $\sigma \geq 0$. On the contrary, if $\mathcal{Q} > 0$ then there exists $\sigma_{\mathcal{T}} > 0$, as defined in Section 5.6 above, such that $\mathcal{T}_2(\sigma,\gamma) > 0$ when $\sigma \in [0,\sigma_{\mathcal{T}})$, $\mathcal{T}_2(\sigma,\gamma) = 0$ when $\sigma = \sigma_{\mathcal{T}}$ and $\mathcal{T}_2(\sigma,\gamma) < 0$ when $\sigma > \sigma_{\mathcal{T}}$. We can show finally that if $\sigma = \bar{\sigma}$ as defined by (51), then

$$\begin{aligned}\mathcal{T}_2(\bar{\sigma},\gamma) &= \tfrac{\sigma a_{11}\left(1-\frac{1}{1-\mathcal{A}}\right)}{(a_{11}a_{00}-a_{10}a_{01})\gamma^2\chi^*\mathcal{N}^2}\left\{\tfrac{a_{00}}{a_{11}a_{00}-a_{10}a_{01}} + \left(\delta+g-\tfrac{dw_1}{dp_1}\right)\right.\\ &- \left.\tfrac{\mathcal{Z}_2+(1+\rho_0)a_{11}g\kappa^*}{a_{11}(\mathcal{Z}_1+\mathcal{Z}_2)}\right\} > 0\end{aligned}$$

We conclude that if $\sigma_{\mathcal{T}} \in (0,+\infty)$ then $\sigma_{\mathcal{T}} > \bar{\sigma}$.

iii) By definition of the line $\Delta_\gamma(\sigma)$ we have $\mathcal{S}_\gamma(\sigma) = \mathcal{D}_2(\sigma,\gamma)/\mathcal{T}_2(\sigma,\gamma)$ and the result follows. □

5.9 Proof of Lemma 5

Consider the expression of $\mathcal{D}(\sigma,\gamma)$ as given by (43). Under Assumption 3, for any given $\sigma \in (0,+\infty)$, the numerator of $\mathcal{D}(\sigma,\gamma)$ is positive for all $\gamma \geq 0$. Consider now the denominator, and more precisely the term between brackets which is positive. We easily derive that:

$$\begin{aligned}\lim_{\gamma\to+\infty}\left[\tfrac{a_{11}}{\gamma\chi^*}\left(1-\tfrac{1}{1-\mathcal{A}}\right)+\tfrac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{\chi^*(1+\rho_0)(1+\rho_1)}\right] &= \tfrac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{\chi^*(1+\rho_0)(1+\rho_1)}\\ \lim_{\gamma\to 0}\left[\tfrac{a_{11}}{\gamma\chi^*}\left(1-\tfrac{1}{1-\mathcal{A}}\right)+\tfrac{a_{01}(\mathcal{Z}_1+\mathcal{Z}_2)}{\chi^*(1+\rho_0)(1+\rho_1)}\right] &= +\infty\end{aligned}$$

Since under Assumption 3, $a_{11}a_{00} - a_{10}a_{01} < 0$, we conclude that the denominator is negative for all $\gamma \geq 0$ if $\sigma \geq \bar{\sigma}$, with $\bar{\sigma}$ the bound defined by (51) and which has been obtained in the case with inelastic labor, i.e. $\gamma = +\infty$. It follows that when $\sigma \geq \bar{\sigma}$, $\mathcal{D}(\sigma, \gamma) < 0$ for all $\gamma \geq 0$. On the contrary, when $\sigma \in [0, \bar{\sigma})$, we get:

$$\lim_{\gamma \to +\infty} \mathcal{D}(\sigma, \gamma) > 0, \quad \lim_{\gamma \to 0} \mathcal{D}(\sigma, \gamma) < 0$$

Since Lemma 4 implies that $\mathcal{D}(\sigma, \gamma)$ is a monotone decreasing function of γ when $\sigma \in [0, \bar{\sigma})$, the results *ii)* are proved.

Notice that the critical bound $\underline{\gamma}(\sigma)$ is defined from expression (43) in Appendix 5.2 as the value of γ such that the denominator of $\mathcal{D}(\sigma, \gamma)$ is equal to zero, and is such that

$$\underline{\gamma}(\sigma) = \frac{a_{11}\left(1 - \frac{1}{1-\mathcal{A}}\right)\sigma}{\chi^*(a_{10}a_{01} - a_{11}a_{00}) - \sigma \frac{a_{01}(\mathcal{Z}_1 + \mathcal{Z}_2)}{(1+\rho_0)(1+\rho_1)}} \tag{54}$$

Then we easily get the derivative

$$\underline{\gamma}'(\sigma) = \frac{a_{11}\left(1 - \frac{1}{1-\mathcal{A}}\right)\chi^*(a_{10}a_{01} - a_{11}a_{00})}{\left[\chi^*(a_{10}a_{01} - a_{11}a_{00}) - \sigma \frac{a_{01}(\mathcal{Z}_1 + \mathcal{Z}_2)}{(1+\rho_0)(1+\rho_1)}\right]^2} \tag{55}$$

which is positive as the consumption good is capital intensive at the private level. □

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